Limit theorems for continuous-state branching processes with immigration

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Definition (branching property and CSBP)

A non-negative Markov process $(X_t(x), t \geq 0)$ is a CSBP if for any $x, y \in \mathbb{R}_+$,

$$X_t(x+y) \overset{d}{=} X_t(x) + \tilde{X}_t(y)$$

where $(\tilde{X}_t(y), t \geq 0)$ is an independent copy of $(X_t(y), t \geq 0)$.

This ensures the existence of a map $t \to v_t(\lambda)$ s.t.

$$\mathbb{E}[e^{-\lambda X_t(x)}] = \exp(-xv_t(\lambda)) \text{ and } v_{s+t}(\lambda) = v_s \circ v_t(\lambda).$$

Theorem (Characterization: Jirina (1958), Lamperti (1967))

$t \mapsto v_t(\lambda)$ is the unique solution to the differential equation

$$\frac{\partial}{\partial t} v_t(\lambda) = -\Psi(v_t(\lambda)), \quad v_0(\lambda) = \lambda.$$

where $\rho := \inf\{z > 0; \Psi(z) \geq 0\}$ is the largest positive root of a Lévy-Khintchine function

$$\Psi(q) = \frac{\sigma^2}{2} q^2 - \beta q + \int_0^\infty \left(e^{-qx} - 1 + qx1_{x\leq 1}\right) \pi(dx)$$

Definition

Consider Galton-Watson branching processes defined inductively by

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{(n)}, \quad Z_0 = 1,$$

where $\xi_i^{(n)}$= the number of children of i at generation n (i.i.d.) and Z_n = the number of particles at generation n.

- $\mathbb{E}[\xi_1^{(1)}] = 1$ (critical), $\mathrm{Var}(\xi_1^{(1)}) = \sigma^2 < \infty$.

Theorem

(a) (Kolmogorov (1938)) $\mathbb{P}(Z_n > 0) \sim 2/n\sigma^2$ as $n \to \infty$.

(b) (Yaglom (1947)) $\mathbb{P}(Z_n/n \in \cdot \,|\, Z_n > 0) \overset{w}{\to} e$, where e is exponential with mean $\sigma^2/2$.

- Consider a sequence of critical GW branching processes $\{Z_n^{(n)} : n \in \mathbb{N}\}$ with initial conditions $Z_0^{(n)}$ satisfying $Z_0^{(n)}/n \to x$. Define

$$X_t^{(n)} = Z_{[nt]}^{(n)}/n.$$

Then $X_t^{(n)}$ converges weakly to a Poisson sum of independent exponential masses, denoted by $X_t(x)$.

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Feller's Theorem: CSBPs

- $\mathbb{E}[e^{-\lambda X_t(x)}] = \exp(-xv_t(\lambda))$ and $v_t(\lambda) = \frac{\lambda}{1+\frac{\sigma^2\lambda t}{2}}$.

Theorem (Feller (1931, 1951))

$X^{(n)} \overset{w}{\Rightarrow} X$ in $D(\mathbb{R}_+)$, where X is the unique solution of

$$X_t(x) = x + \sigma \int_0^t \sqrt{X_s(x)}dB_s$$

where B is one-dimensional Brownian motion.

Theorem (Dawson-Li (2012))

$$X_t(x) = x + \sigma \int_0^t \int_0^{X_s(x)} W(ds, du) + \int_0^t \int_0^\infty \int_0^{X_{s-}(x)} z\tilde{N}_0(ds, dz, du),$$

where $W(ds, du)$ white noise on $\mathbb{R}_+^2$ with intensity $dsdu$, and $\widetilde{N}(ds, dz, du)$ compensated Poisson random measure on $\mathbb{R}_+^3$ with intensity $ds\pi(dz)du$

CSBP with immigration: CBI

- Laplace exponent of subordinator:

$$\Phi(q) \quad = \beta q + \int_0^\infty (1 - e^{-qu})\nu(du)$$

- Laplace exponent of a a spectrally positive Lévy process with finite mean:

$$\Psi(q) \quad = bq + \tfrac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\pi(du)$$

where $\int_0^\infty (u \wedge u^2)\pi(du) < \infty$.

Theorem (Kawazu-Watanbe (1971))

A CBI process with branching and immigration mechanisms Ψ and Φ, is a strong Markov process $(Y_t, t \geq 0)$ taking values in $[0, \infty)$ whose transition kernels are characterized by

$$\mathbb{E}_x[e^{-\lambda Y_t}] = \exp\left(-xv_t(\lambda) - \int_0^t \Phi(v_s(\lambda))ds\right)$$

Theorem (Emilia Caballero et al. 2013)

A CBI (Ψ, Φ) process with initial value x, denoted by Y_t, solving the functional equation

$$Y_t = x + \xi_{\int_0^t Y_s ds} + \eta_t$$

where ξ_t is a spectrally positive Lévy process with Laplace exponent Ψ and η_t is a Lévy subordinator with Laplace exponent Φ.

The Alpha-CIR model

Consider a special CBI process, where $\Psi(q) = aq + \frac{\sigma^2}{2}q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}q^\alpha$ and $\Phi(q) = abq$, given by

$$dV_t = a(b - V_t)dt + \sigma\sqrt{V_t}dB_t + \sigma_Z \sqrt[\alpha]{V_t}dZ_t$$

where

- $B = (B_t, t \geq 0)$ a Browinan motion
- $Z = (Z_t, t \geq 0)$ a spectrally positive α-stable compensate Lévy process with parameter $\alpha \in (1, 2]$

- Pathwise uniqueness of SDE, Fu and Li (SPA, 2010)
- The case of $\alpha = 2$, Cox-Ingersoll-Ross model (Econometrica, 1985).

The Alpha-CIR as interest rate model

- Current sovereign bond markets with persistency of low interest rates and significant fluctuations at local extent.

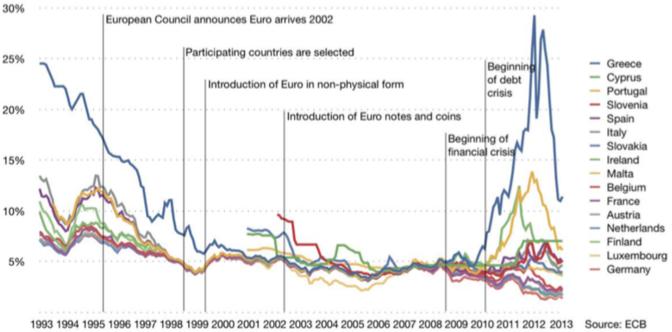

Figure: 10 years interest rates of Euro area countries.

The zero-coupon price

Consider a zero-coupon bond of maturity T at time $t \leq T$

$$B(t,T) = \mathbb{E}\Big[\exp\Big\{-\int_t^T V_s ds\Big\}\Big|\mathcal{F}_t\Big]$$

Proposition (Jiao-M.-Scotti, Finance Stoch., 2017)

(a) The bond price $B(0,T)$ is decreasing with respect to α.

(b) J_t^y the number of jumps of V with jump size larger than y in $[0,t]$,

$$\mathbb{E}\big[e^{-pJ_t^y}\big] = \exp\Big(-l(p,y,t)r_0 - ab\int_0^t l(p,y,s)ds\Big)$$

where $l(p,y,t)$ is the unique solution of the following equation

$$\frac{\partial l(p,y,t)}{\partial t} = \sigma_Z^\alpha \int_y^\infty \big(1-e^{-p-l(p,y,t)\zeta}\big)\mu_\alpha(d\zeta) - \Psi^{(y)}(l(p,y,t)),$$

with initial condition $l(p,y,0) = 0$

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Conclusion: interpret the low interest

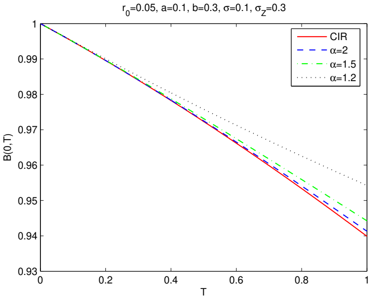


Figure: Bond price is decreasing w.r.t. α, which inversely related to the tail fatness. curve CIR (in red) corresponds to $\sigma_Z = 0$

- The expected (first) Large jump time is increasing with α

The Alpha-CIR as stochastic volatility

Consider the Alpha-Heston model:

$$\begin{aligned} dS_t &= S_t(rdt + \sqrt{V_t}dW_t) \\ dV_t &= a(b - V_t)dt + \sigma\sqrt{V_t}dB_t + \sigma_Z \sqrt[\alpha]{V_t}dZ_t \end{aligned}$$

where V follows the α-CIR model.

- The case of $\alpha = 2$, Heston stochastic volatility model (Review of Financial Studies, 1993)

Implied volatility at extreme strikes for VIX option

Consider $\Sigma_{\text{VIX}}(T,k)$ be the implied volatility of call options written on VIX with maturity T and strike $K = e^k$

Proposition (Jiao-M.-Scotti-Zhou, Math. Finance, 2021)

The right wing of $\Sigma_{\text{VIX}}(T,k)$ has the following asymptotic shape:

$$\Sigma_{\text{VIX}}(T,k) \sim \left(\frac{\psi(2\alpha)}{T}\right)^{1/2} \sqrt{k}, \quad k \to +\infty.$$

where

$$\psi(q) = 2 - 4(\sqrt{q^2 + q} - q)$$

Implied volatility at extreme strikes for VIX option

Proposition (continued)

The left wing of $\Sigma_{\text{VIX}}(T,k)$ has the following asymptotic shape as $k \downarrow \frac{1}{2}\log B(\Delta)$:

1. if $\sigma > 0$, then $\Sigma^2_{\text{VIX}}(T,k) \sim D_\sigma \left(-\log\left(e^k - \sqrt{B(\Delta)}\right)\right)^{-1}$,
2. if $\sigma = 0$, then $\Sigma^2_{\text{VIX}}(T,k) \sim D_0 \left(e^k - \sqrt{B(\Delta)}\right)^{\frac{2-\alpha}{\alpha-1}}$,

Implied volatility: an upward-sloping smile

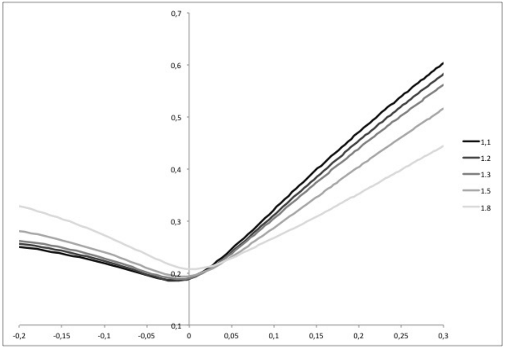
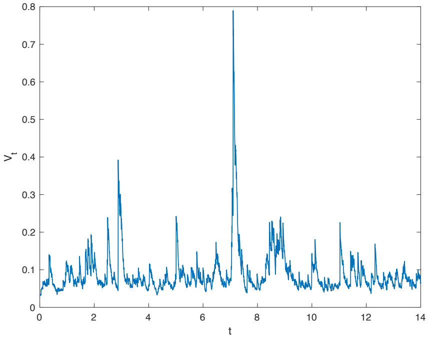


Figure: The implied volatility curves of the VIX options for different values of α with $a = 5$, $b = 0.144$, $\sigma = 0.25$, $\sigma_N = 0.3$, $\rho = 0$, and $T = 0.25$

- Implied volatility of VIX options for the Heston model given by Nicolato *et al.* (2017): downward sloping!

The simulation of the VIX

The following Figure provides a simulation of the variance process V for a period of $T = 14$, in comparison with the empirical VIX data (from 2004 to 2017). The parameters: $a = 5$, $b = 0.14$, $\sigma = 0.08$, $\sigma_Z = 1$ and $\alpha = 1.26$.

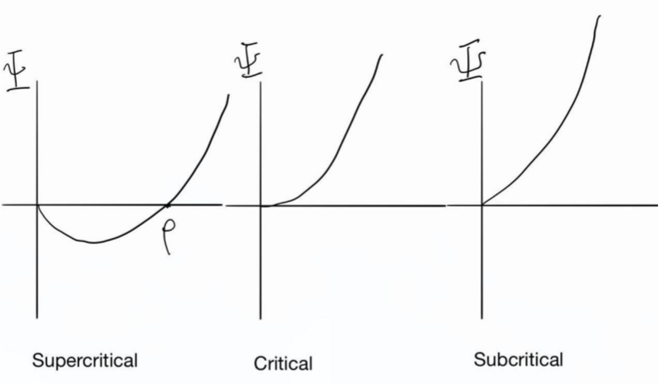


- Jiao-M.-Scotti-Sgarra (Energy Economics, 2019): power price on Italian market (2004-2015), $\alpha = 1.5$.

Three categories

Define $b := \Psi'(0+)$.

- $b \in (-\infty, 0)$: supercritical case
- $b = 0$: critical case
- $b \in (0, \infty)$: subcritical case



Asymptotic behaviors: known

For any $t \geq 0$, let $\lambda \mapsto \eta_t(\lambda)$ be the inverse map of $\lambda \mapsto v_t(\lambda)$.

Theorem (Pinsky (1972), Li (2010))

Consider a conservative CBI process $(Y_t, t \geq 0)$.

	$\int_0 \frac{\Phi(u)}{\lvert\Psi(u)\rvert} du < \infty$	$\int_0 \frac{\Phi(u)}{\lvert\Psi(u)\rvert} du = \infty$
$b < 0$	$\eta_t(\lambda) Y_t \xrightarrow{d}$ proper	$\eta_t(\lambda) Y_t \xrightarrow{p} \infty$
$b \geq 0$	$Y_t \xrightarrow{d}$ proper	$Y_t \xrightarrow{p} \infty$

- In the non-critical case, the condition $\int_0 \frac{\Phi(u)}{|\Psi(u)|} du < \infty$ is equivalent to $\int^\infty \ln(u)\nu(du) < \infty$ where ν is the immigration measure.

Asymptotic behaviors: $\int_0 \frac{\Phi(u)}{|\Psi(u)|}du < \infty$

Theorem (Foucart-M.-Yuan (2020+))

Consider a super-critical CBI(Ψ, Φ) process. Let $0 < \lambda < \rho$. Then, $\eta_t(\lambda)Y_t \underset{t\to\infty}{\longrightarrow} W^\lambda$ $\mathbb{P}_x$-a.s. where W^λ is a non-degenerate proper random variable with Laplace exponent

$$\mathbb{E}_x[e^{-\theta W^\lambda}] = \exp\left(-xv_{-\ln\theta/b}(\lambda) + \int_0^{v_{-\ln\theta/b}(\lambda)} \frac{\Phi(u)}{\Psi(u)}du\right)$$

- If $\int^\infty (x\ln x)\pi(dx) < \infty$ then $\eta_t(\lambda) \underset{t\to\infty}{\sim} K_\lambda e^{bt}$ for some constant $K_\lambda > 0$, where π is the branching measure.

Theorem (Li-M. (2015))

Consider a sub-critical CBI(Ψ, Φ) process with Grey's condition. If $\int_1^\infty u^\delta \nu(du) < \infty$, then it is exponentially ergodic.

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Asymptotic behaviors: $\int_0 \frac{\Phi(u)}{|\Psi(u)|}du = \infty$

Theorem (Foucart-M.-Yuan (2020+))

Let $(Y_t, t \geq 0)$ be a supercritical CBI(Ψ, Φ). Assume $\int_0 \frac{\Phi(u)}{|\Psi(u)|}du = \infty$. Then, there exists no deterministic renormalization function $(\eta(t), t \geq 0)$ such that $\eta(t)Y_t \underset{t\to\infty}{\longrightarrow} V$ almost-surely for some non-degenerate random variable V.

Theorem (Duhalde-Foucart-M. (2014))

A (sub)critical CBI(Ψ, Φ) process is recurrent or transient according as

$$\mathcal{E} := \int_0 \frac{dx}{\Psi(x)} \exp\left(-\int_x^1 \frac{\Phi(u)}{\Psi(u)}du\right) = \infty \text{ or } < +\infty.$$

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To understand the paper of Pinsky (1972) published in Bulletin of The American Mathematical Society, which has no proof for any result presented.

THEOREM 2. *Let* $X = (x_t, P_x)$ *be a conservative CBI process with* $-\infty < \rho < 0$. *For* $x > 0$ *let*

$$H(x) = \int_{e^{-x}}^{1} \frac{F(u)}{R(u)} du, \quad m(x) = \exp(H(\log x)).$$

Assume that as $x \to \infty$, *we have*

(C1) $$H(x) \to \infty,$$

(C2) $$xH'(x) \to 0.$$

Then for $0 \leqq u \leqq 1$,

(*) $$P_x\{m(x_t)/m(e^{ct}) \leqq u\} \to u^{1/c},$$

as $t \to \infty$, *here* $c = -\rho < 0$.

Weak law: $\int_0 \frac{\Phi(u)}{|\Psi(u)|}du = \infty$

Theorem (Foucart, M., Yuan (2020+))

Let $(Y_t, t \geq 0)$ be a non-critical CBI(Ψ, Φ). Then, for all $x \geq 0$, we have

$$r_t(1/Y_t) := \int_{v_t(1/Y_t)}^{1/Y_t} \frac{\Phi(u)}{\Psi(u)}du \xrightarrow{d} \mathrm{e}_1, \text{ as } t \to +\infty \text{ under } \mathbb{P}_x$$

where e_1 is an exponential random variable with parameter 1.

Corollary

Assume $\int_0 \frac{\Phi(u)}{|\Psi(u)|}du = \infty$ and that the process is non-critical. Let $(Y_t, t \geq 0)$ and $(\tilde{Y}_t, t \geq 0)$ be two independent CBI(Ψ, Φ) processes started from 0. Then

$$\mathbb{P}(Y_t/\tilde{Y}_t \underset{t\to\infty}{\longrightarrow} 0) = \mathbb{P}(Y_t/\tilde{Y}_t \underset{t\to\infty}{\longrightarrow} \infty) = \frac{1}{2}.$$

Weak law: further development

Fix λ_0 such that $\lambda_0 \in (0, +\infty)$ in the (sub)critical case and $\lambda_0 \in (0, \rho)$ in the supercritical case. Put

$$\varphi(\lambda) = \int_\lambda^{\lambda_0} \frac{du}{|\Psi(u)|}, \quad 0 < \lambda < \lambda_0.$$

The mapping $\varphi : (0, \lambda_0) \to (0, +\infty)$ is strictly decreasing, write g for its inverse mapping, g is a strictly decreasing continuous function on $(0, \infty)$, and

$$\lim_{x\to\infty} g(x) = 0, \quad \lim_{x\to 0} g(x) = \lambda_0.$$

we introduce $H(x)$ to characterize the divergence of the integral:

$$H(x) := \begin{cases} \frac{1}{|b|}\int_{e^{-x}}^{1} \frac{\Phi(u)}{u} du, \text{ if } b \in (-\infty, 0) \cup (0, \infty); \\ \int_{g(x)}^{\lambda_0} \frac{\Phi(u)}{|\Psi(u)|} du, \text{ if } b = 0 \end{cases}, \quad x \geq 0$$

Three conditions

(S) (slow-divergence) $xH'(x) \to 0$ as $x \to +\infty$ and $H(x) \to +\infty$;

(L) (log-divergence) $xH'(x) \to a$ for some constant $a > 0$ as $x \to +\infty$;

(F) (fast-divergence) $xH'(x) \to +\infty$ as $x \to +\infty$ and H' is regularly varying at $+\infty$.

In the non-critical case, in terms of the tail of the immigration measure ν:

(S) (slow-divergence) $\bar{\nu}(x)\ln x \to 0$ as $x \to \infty$ and $\int_1^\infty \frac{\bar{\nu}(x)}{x} = \infty$;

(L) (log-divergence) $\bar{\nu}(x)\ln x \to c$ for some constant $c > 0$ as $x \to \infty$;

(F) (fast-divergence) $\bar{\nu}(x)\ln x \to \infty$ as $x \to \infty$ and $\bar{\nu}$ is slowly varying at ∞.

In the non-critical case, in terms of the tail of the immigration measure ν:

- **(S)** (slow-divergence) $\bar{\nu}(x)\ln x \to 0$ as $x \to \infty$ and $\int_1^\infty \frac{\bar{\nu}(x)}{x} = \infty$;
- **(L)** (log-divergence) $\bar{\nu}(x)\ln x \to c$ for some constant $c > 0$ as $x \to \infty$;
- **(F)** (fast-divergence) $\bar{\nu}(x)\ln x \to \infty$ as $x \to \infty$ and $\bar{\nu}$ is slowly varying at ∞.

Example

In the non-critical case,

1. If $\bar{\nu}(x) \sim \frac{1}{\ln x \ln \ln x}$ as $x \to \infty$, then $H(x) \sim (\ln \ln x)/|b|$ and $H'(x) \sim 1/(|b| x \ln x)$. Condition (S) is satisfied.
2. If $\bar{\nu}(x) \sim c/\ln x$ for some constant $c > 0$, as $x \to \infty$, then $H'(x) \sim c/(|b|x)$. Condition (L) is satisfied.
3. If $\bar{\nu}(x) \sim \frac{\ln \ln x}{(\ln x)^\delta}$, $\quad (0 < \delta \leq 1)$ as $x \to \infty$, then $H'(x) \sim (x^{-\delta} \ln x)/|b|$. If as $x \to \infty$, $\bar{\nu}(x) \sim 1/(\ln \ln x)$, then $H'(x) \sim 1/(|b| \ln x)$. Both cases satisfy Condition (F).

Non-critical case

Set

$$\rho_t := \begin{cases} 1, \text{ if } b > 0; \\ v_{-t}(\lambda_0), \text{ if } b < 0. \end{cases}$$

Theorem

(i) If Condition (S) holds, let $m(x) := \exp(\int_{1/x}^1 \frac{\Phi(u)}{\Psi(u)} du)$ for $x > 0$. Then

$$\frac{\ln \rho_t Y_t}{t} \xrightarrow{p} 0 \quad \text{and} \quad m(\rho_t Y_t)/m(e^{|b|t}) \xrightarrow{d} U, \text{ as } t \to \infty, \tag{1}$$

where U is uniformly distributed on $[0, 1]$.

Non-critical case

Theorem (continued)

(ii) If Condition (L) holds, then

$$\frac{\ln \rho_t Y_t}{t} \xrightarrow{d} |b| U_L, \text{ as } t \to \infty, \tag{2}$$

where $\mathbb{P}(U_L \le \lambda) = \left(\frac{\lambda}{1+\lambda}\right)^a, \quad \lambda \ge 0.$

Theorem (continued)

(iii) If Condition (F) holds with $0 \le \delta \le 1$, then

$$\frac{\ln Y_t}{t} \xrightarrow{p} \infty \quad \text{and} \quad t\Phi(1/Y_t) \xrightarrow{d} \mathrm{e}_1, \text{ as } t \to \infty.$$

In particular, if $0 < \delta \le 1$, then we have

$$h(t) = t^{1/\delta} L^*(t) \quad \text{and} \quad \frac{\ln Y_t}{h(|b|t)} \xrightarrow{d} U_F, \text{ as } t \to \infty, \tag{3}$$

where L^* is some slowly varying function at ∞ and U_F follows the extreme distribution given by $\mathbb{P}(U_F \le \lambda) = \exp(-1/\lambda^\delta), \quad \lambda \ge 0$.

Proposition

Let $(\eta_t, t \geq 0)$ be a subordinator with Laplace exponent Φ. Assume that Φ is slowly varying at 0, then

$$t\Phi(1/\eta_t) \xrightarrow{d} \mathrm{e}_1 \text{ as } t \to \infty.$$

A small correction for Pinsky's result

THEOREM 2. *Let $X = (x_t, P_x)$ be a conservative CBI process with $-\infty < \rho < 0$. For $x > 0$ let*

$$H(x) = \int_{e^{-x}}^{1} \frac{F(u)}{R(u)}\,du, \quad m(x) = \exp(H(\log x)).$$

Assume that as $x \to \infty$, we have

(C1) $$H(x) \to \infty,$$

(C2) $$xH'(x) \to 0.$$

Then for $0 \leqq u \leqq 1$,

$\{ \quad \geq u^{-1}\} \to u$

(*) $$P_x\{m(x_t)/m(e^{ct}) \leqq u\} \to u^{1/c},$$

as $t \to \infty$, here $c = -\rho < 0$.

Critical case

Suppose that π satisfies

$$\bar{\pi}(u) \underset{u\to\infty}{\sim} -\frac{1}{\Gamma(-\alpha)} u^{-1-\alpha} \ell(u),$$

where $\bar{\pi}(u) = \pi(u, \infty)$ for $u > 0$, $0 < \alpha < 1$ and ℓ is slowly varying at ∞.

Theorem

(i) If Condition (S) holds, then

$$\frac{m(Y_t)}{m(1/g(t))} \xrightarrow{d} V \text{ as } t \to \infty,$$

where V is uniformly distributed on $[0, 1]$.
(ii) If Condition (L) holds, then

$$g(t)Y_t \xrightarrow{d} V_L \text{ as } t \to \infty,$$

where $\mathbb{E}[e^{-\lambda V_L}] = (1 + \lambda^{\alpha})^{-a}, \forall \lambda \geq 0.$

Theorem (continued)

(iii) If Condition (F) holds with $\delta > 0$, then

$$\varrho_t Y_t \xrightarrow{d} V_F \text{ as } t \to \infty \tag{4}$$

where $\mathbb{E}[e^{-\theta V_F}] = \exp(-\theta^{\delta\alpha})$, for all $\theta \geq 0$, with $\varrho_t = \Phi^{-1}(1/t) = t^{-1/(\delta\alpha)}\bar{\ell}(t)$ as $t \to \infty$ for some slowly varying function $\bar{\ell}$ at ∞.

In fact, (4) is equivalent to

$$t\Phi(1/Y_t) \xrightarrow{d} V_F^{-\delta\alpha}.$$

If Condition (F) holds with $\delta = 0$, then $t\Phi(1/Y_t) \xrightarrow{d} \mathrm{e}_1$ as $t \to \infty$.

Thank you for your attention!